

Correlations of photon trajectories in the problem of light scintillations

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Abstract

A distribution function approach is applied to describe the dynamics of the laser beam in the Earth atmosphere. Using a formal solution of the kinetic equation for the distribution function, we have developed an iterative scheme for calculation of the scintillation index (σ^2). The problem reduces to obtaining the photon trajectories and their correlations. Bringing together theoretical calculations and many-fold computer integrations, the value of σ^2 is obtained. It is shown that a considerable growth of σ^2 in the range of a moderate turbulence is due to the correlations of different trajectories. The criteria of applicability of our approach for both the coherent and partially coherent light are derived.

1 Introduction

Basic principles of the radiation transfer theory were formulated in the seminal paper of Schuster [1] as early as the beginning of the 20th century. The

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paper [1] was devoted to the light propagation in a foggy atmosphere. Since then the Schuster approach has obtained many applications in such important fields as astronomy, laser communication and radar systems, remote sensing etc.

The range and performance of light communication systems are limited significantly by an unfavorable influence of local fluctuations of the refractive index of the Earth atmosphere. On the other hand, high sensitivity of the photon trajectories to the fluctuations can be used for the atmosphere diagnostics [2]. The key issue about the index-of-refraction structure constant C_n^2 is that it cannot be reliably computed from first principles. The refractive-index fluctuations arise from temperature inhomogeneities of the air. The inhomogeneities cause turbulent eddies which give rise to a random distribution of the air density([3]-[5]). This results in random spatial variations of the refractive index.

The turbulent eddies are described by a wide range of characteristic lengths of inhomogeneities. These lengths cover the interval from few millimeters (the inner radius, l_0) to hundred meters (the outer radius, L_0). Therefore, various types of beam scattering are observed. The scattering by large-size eddies results in random redirections of the beam as a whole. This process is known in the literature as a "wandering" or "dancing" of the beam [6],[7]. On the other hand, the scattering by small-size eddies causes spreading of the beam. For a long-distance propagation or a strong turbulence, the beam radius becomes greater than the characteristic sizes of the inhomogeneities. In this case the probability of the beam to be redirected becomes small and the relative value of the wandering radius decreases [8].

The beam wandering and broadening can be considered as the specific manifestations of a more general phenomenon, namely the intensity fluctuations (i.e. scintillations) caused by the atmosphere turbulence. The scintillations have a tendency of saturating for a long-distance propagation [9],[10] (the regime of a strong turbulence). This is because in the course of propagation the radiation acquires the properties of the Gaussian statistics when the signal-to-noise ratio (SNR) tends to unity. The asymptotic behavior of the scintillation index, $\sigma^2 \rightarrow 1$, was explained in Refs. [11]-[13]. Moreover, it was shown quite generally that this property stays unchanged for any refractive index distribution, provided the response time of the recording instrument is short compared with the source coherence time. This result was confirmed analytically in [14].

At the same time, calculations, performed by different methods in [15]

and [16], show a possibility of significant suppression of the scintillations. To this end partially coherent laser beams with the coherence time shorter than the detector integration time (a slow detector) can be used. The case of a partial coherence was also studied in [17],[18]. Recent theoretical and experimental developments on propagation of partially coherent beams in a turbulent atmosphere were discussed in [19].

There are several analytical approaches explaining behavior of the scintillation index in the case of strong turbulence [16, 20, 21]. Their analysis is based on the physical picture where four waves, forming the second moment of the intensity, conserve only pair correlations in course of long-distance propagation. Two different pairs of the photon trajectories contribute into the square of the photon density at the detector. Dashen used the Feynman path integrals to prove that in a convincing manner [20].

The recent interest to beam propagation was awakened by the development of quantum communication in the free atmosphere [22], [23]. Detailed studies of the effect of the turbulence-induced losses on the quantum state of the light in the course of satellite-mediated communication and for realization of the entanglement transfer in the atmosphere were reported in Refs. [24] and [25].

The formalism of the photon distribution function (the photon density in the coordinate-momentum space [26]), is also applicable to the problem of scintillations [8, 16, 27, 28]. The mentioned papers are based on a physical picture, which is similar to the described above. The method of photon distribution function is used for description of both the classical and the quantum light including propagation of single-photon pulses (see, for example, Refs. [27, 29, 30]). Solution of the kinetic equation for the operator of photon density is based on the method of characteristics. The assumption of weak disturbances of photon momenta by the atmosphere (the paraxial approximation) reduces the problems of scintillations to the problem of obtaining photon trajectories and their correlations. A slowly varying fluctuating force, deflecting photon trajectories from straight lines, describes the effect of the atmospheric eddies.

In this work we study the scintillation index for moderate and strong turbulences, when correlation of trajectories of only two photons is required. Accuracy of the calculations depends on the accuracy of obtaining the trajectories. Using high-order iterations and bringing together analytical and numerical procedures, we calculate the scintillation index. Our main interest is to analyze the range of moderate turbulence strengths where previous the-

ories do not ensure a reliable description. Comparison of the obtained results with those represented in [16] helps indicate the range of turbulence where a simplified approach should be corrected by high-order iterations. Also, our studies describe more realistically the effect of partial coherence.

2 Photon distribution function approach

The photon distribution function is defined by analogy with distribution functions in solid state physics. In particular, it is similar to the phonon distribution function. Both of them are defined as [26],[31]

$$f(\mathbf{r}, \mathbf{q}, t) = \frac{1}{V} \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}} b_{\mathbf{q}+\mathbf{k}/2}^\dagger b_{\mathbf{q}-\mathbf{k}/2}, \quad (1)$$

where $b_{\mathbf{q}}^\dagger$ and $b_{\mathbf{q}}$ are the bosonic creation and annihilation operators of photons or phonons with the momenta \mathbf{q} , and $V \equiv L_x L_y L_z$ is the normalizing volume. Polarization of the corresponding modes is not specified in (1). In the paraxial approximation, assumed here, the initial polarization of the beam remains almost unchanged even for a long-distance propagation (see, for example, Ref. [32]).

The operator $f(\mathbf{r}, \mathbf{q}, t)$ describes the photon (phonon) density in the phase (\mathbf{r}, \mathbf{q}) space. Usually, the characteristic sizes of spatial inhomogeneities of the radiation field are much greater than the wave-length. In this case the sum in Eq. (2) can be restricted by small k . Here and in what follows we consider that $k < k_0 \ll q_0$, where q_0 is the wave vector corresponding to the central frequency of the radiation, $\omega_0 = cq_0$. At the same time k_0 should be taken sufficiently large to provide a required accuracy of the beam profile description.

The evolution of the Heisenberg operator $f(\mathbf{r}, \mathbf{q}, t)$ is determined by the commutator

$$\partial_t f(\mathbf{r}, \mathbf{q}, t) = \frac{1}{i\hbar} [f(\mathbf{r}, \mathbf{q}, t), H], \quad (2)$$

where

$$H = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} - \sum_{\mathbf{q}, \mathbf{k}} \hbar\omega_{\mathbf{q}} n_{\mathbf{k}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}+\mathbf{k}}, \quad (3)$$

is the Hamiltonian of photons in a medium with a fluctuating refractive index $n(\mathbf{r})$ ($n_{\mathbf{k}}$ is its Fourier transform), $\hbar\omega_{\mathbf{q}} = \hbar cq$ and $\mathbf{c}_{\mathbf{q}} = \frac{\partial\omega}{\partial\mathbf{q}}$ are the vacuum values of the photon energy and velocity, respectively.

Assuming the characteristic values of the photon momentum to be much greater than the wave vectors of turbulence, the kinetic equation for the photon distribution function can be written as

$$\{\partial_t + \mathbf{c}_q \partial_{\mathbf{r}} + \mathbf{F}(\mathbf{r}) \partial_{\mathbf{q}}\} f(\mathbf{r}, \mathbf{q}, t) = 0, \quad (4)$$

where $\mathbf{F}(\mathbf{r}) = \omega_0 \partial_{\mathbf{r}} n(\mathbf{r})$ is the random force originating from the atmospheric turbulence. The general solution of Eq. (4) is given by

$$f(\mathbf{r}, \mathbf{q}, t) = \phi \left\{ \mathbf{r} - \int_0^t dt' \frac{\partial \mathbf{r}(t')}{\partial t'}; \mathbf{q} - \int_0^t dt' \frac{\partial \mathbf{q}(t')}{\partial t'} \right\}, \quad (5)$$

where the function $\phi(\mathbf{r}, \mathbf{q})$ is the "initial" value of $f(\mathbf{r}, \mathbf{q}, t)$, i.e.

$$\phi(\mathbf{r}, \mathbf{q}) = \frac{1}{V} \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}} (b_{\mathbf{q}+\mathbf{k}/2}^+ b_{\mathbf{q}-\mathbf{k}/2})|_{t=0} \equiv \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}} \phi(\mathbf{k}, \mathbf{q}). \quad (6)$$

The derivatives $\frac{\partial \mathbf{r}(t')}{\partial t'}$ and $\frac{\partial \mathbf{q}(t')}{\partial t'}$ should satisfy the equations

$$\begin{aligned} \frac{\partial \mathbf{r}(t')}{\partial t'} &= \mathbf{c}[\mathbf{q}(t')] \\ \frac{\partial \mathbf{q}(t')}{\partial t'} &= \mathbf{F}[\mathbf{r}(t')], \end{aligned} \quad (7)$$

completed with the boundary conditions $\mathbf{r}(t') = \mathbf{r}$ and $\mathbf{q}(t') = \mathbf{q}$ for $t = t'$. As we see, Eqs. (7) coincide with the classical (the Newton) equations of motion of a point particle moving with the velocity \mathbf{c}_q and affected by an external force $\mathbf{F}(\mathbf{r})$. Formal solutions of Eqs. (7) can be written as

$$\mathbf{q}(t') = \mathbf{q} + \int_t^{t'} dt'' \mathbf{F}[\mathbf{r}(\mathbf{q}, t'')] \quad (8)$$

and

$$\mathbf{r}(t') = \mathbf{r} - \mathbf{c}_q(t - t') - \frac{c}{q_0} \int_t^{t'} dt'' (t'' - t') \mathbf{F}[\mathbf{r}(t'')], \quad (9)$$

Eqs. (8) and (9) allow us to rewrite the expression (5) as

$$f(\mathbf{r}, \mathbf{q}, t) = \phi \left\{ \mathbf{r} - \mathbf{c}_q t + \frac{c}{q_0} \int_0^t dt' t' \mathbf{F}[\mathbf{r}(\mathbf{q}, t')]; \mathbf{q} - \int_0^t dt' \mathbf{F}[\mathbf{r}(\mathbf{q}, t')] \right\}. \quad (10)$$

If $\mathbf{F}(\mathbf{r})$ is a known function, an approximate value for $f(\mathbf{r}, \mathbf{q}, t)$ can be obtained by inserting the term $\mathbf{r}(\mathbf{q}, t') \approx \mathbf{r} - \mathbf{c}_q(t - t')$ into Eq. (10). In this case the argument of the fluctuating force $\mathbf{F}[\mathbf{r}(\mathbf{q}, t')]$ is replaced by a straight line, that is correct only in the absence of the turbulence. Improvement of the theory can be achieved if the argument of \mathbf{F} accounts for the turbulence.

It follows from Eq. (10) that statistical properties of the radiation depend not only on the turbulence but also on the initial distribution function $\phi(\mathbf{r}, \mathbf{q})$. This function is determined by the source field. Its explicit form is determined in the course of "sewing" of the near-aperture and the atmospheric fields [16] given by the amplitudes $b_q(b_q^\dagger)$. We consider the light propagation in the z -direction. The source field is assumed to be described by the Gaussian function, $\Phi(\mathbf{r}) = (2/\pi)^{1/2} \frac{1}{r_0} e^{-r_\perp^2/r_0^2}$. Then the propagating amplitudes are given by

$$b_{\mathbf{q}_\perp, q_0}(t=0) = b(2\pi/S)^{1/2} r_0 e^{-q_\perp^2 r_0^2/4}, \quad (11)$$

where b is the near-aperture amplitude of the laser field, index (\perp) means the perpendicular to the z -axis components, and $S = L_x L_y$.

We will take into account the effect of the phase diffuser by multiplying the distribution $\Phi(\mathbf{r})$ by the phase factor $e^{-i\mathbf{a}\mathbf{r}_\perp}$ where the quantity \mathbf{a} is a random variable. In this case Eq. (11) should be modified by substituting in its right-hand side $\mathbf{q}_\perp + \mathbf{a} \equiv \mathbf{q}_a$ for \mathbf{q}_\perp . Such a simple modeling of the phase diffuser is justified if (i) the detection time is much longer than the characteristic time of the variation of \mathbf{a} (slow detector) and (ii) there is a large root mean square of the phase fluctuations. (More detailed analysis is presented in [28].) This case corresponds to the Gaussian distribution of \mathbf{a} :

$$P(a_{x,y}) = \frac{\lambda}{2\pi^{1/2}} e^{-a_{x,y}^2 \lambda^2/4}, \quad (12)$$

with a covariance $\langle a_{x,y}^2 \rangle = \lambda^{-2}$ and the transverse correlation function of the outgoing field (at $t=0$) is given by

$$\langle E(\mathbf{r}_\perp) E(\mathbf{r}_\perp + \mathbf{\Delta}) \rangle_a = E_0^2 e^{-[r_\perp^2 + (\mathbf{r}_\perp + \mathbf{\Delta})^2] r_0^{-2}} e^{-\Delta^2 \lambda^{-2}}. \quad (13)$$

Here $E_0 = E(r_\perp = 0, \Delta = 0, t = 0)$ and the notation $\langle \dots \rangle_a$ means averaging over distribution $P(a_{x,y})$. The radiation, whose correlation properties are described by function (13), is referred to as the Gaussian Shell-model field. The parameter λ in the exponential factor describes the decrease of the transverse correlation length. It can also be said that this parameter generates a new characteristic length, $1/r_1$, in the momentum distribution

(i.e., in the \mathbf{q} -domain). This is seen from the explicit term for $\phi(\mathbf{k}, \mathbf{q})$ which after averaging over the fluctuations of \mathbf{a} reduces to

$$\langle \phi(\mathbf{k}, \mathbf{q}) \rangle_{\mathbf{a}} = 2\pi \frac{b^\dagger b}{VS} r_1^2 e^{-q_\perp^2 \frac{r_1^2}{2} - k_\perp^2 \frac{r_0^2}{8}}, \quad (14)$$

where $r_1^2 = r_0^2 \left(1 + 2r_0^2 \lambda^{-2}\right)^{-1}$, and variables q_z and k_z are omitted.

It is seen from Eq. (14) that q_\perp is distributed in the range of the order of $\sqrt{2}/r_1$ that is greater than the one for coherent beam. In contrast, the characteristic value of \tilde{k} depends only on the initial size of the beam ($\tilde{k} \sim \sqrt{8}/r_0$).

In the course of light propagation, the diffraction phenomena and scattering by atmospheric inhomogeneities broaden the beam resulting in decrease of \tilde{k} . At the same time, the value of \tilde{q} increases with the distance. This is because of the Brownian-like motion of photons in the \mathbf{q}_\perp -domain (see Ref. [16]). Such a simple physical picture, elucidating evolution of the beam geometry, is, however, not applicable to the description of scintillations. The phenomenon of scintillations is more complicated and can be described in terms of spatio-temporal correlations of four waves.

3 Scintillation index

The photon distribution function is used here to obtain the scintillation index σ^2 . The definition of σ^2 is given by

$$\sigma^2 = \frac{\langle I^2(\mathbf{r}) \rangle - \langle I(\mathbf{r}) \rangle^2}{\langle I(\mathbf{r}) \rangle^2}. \quad (15)$$

The photon density $I(\mathbf{r}, t)$ is expressed in terms of the distribution function as

$$I(\mathbf{r}, t) = \sum_{\mathbf{q}} f(\mathbf{r}, \mathbf{q}, t) = 2\pi \frac{b^\dagger b r_0^2}{SV} \sum_{\mathbf{q}, \mathbf{k}} e^{-i\mathbf{k}[\mathbf{r} - \mathbf{c}(\mathbf{q})t + \frac{c}{q_0} \int_0^t dt' t' \mathbf{F}(\mathbf{r}(\mathbf{q}, t'))] - Q_{\mathbf{a}}^2 \frac{r_0^2}{2} - k^2 \frac{r_0^2}{8}}, \quad (16)$$

where $\mathbf{Q}_{\mathbf{a}} \equiv \mathbf{Q} + \mathbf{a} = \mathbf{q} + \mathbf{a} - \int_0^t dt' \mathbf{F}[\mathbf{r}(\mathbf{q}, t')]$. The summation is taking over \mathbf{q}_\perp and \mathbf{k}_\perp components, while q_z and k_z are considered to be fixed: $q_z = q_0$ and $k_z = 0$. The exponential term originates from the solution (10) of the kinetic equation (4).

To obtain $\langle I(\mathbf{r}, t) \rangle$, three independent averagings are required. One of them concerns the source variables. In the case of a coherent state of the source, $|\beta\rangle$, we have $\langle b^\dagger b \rangle = |\beta|^2$. The second averaging over a random phase of the diffuser should be carried out as explained by Eq. (14). The third averaging deals with the fluctuating force \mathbf{F} . These three actions can be performed independently that facilitates the analysis. Also, the calculations are simplified if we use the identity

$$e^{-Q^2 r_0^2/2} \equiv \int \frac{d\mathbf{p}}{2\pi r_0^2} e^{i\mathbf{p}\mathbf{Q} - p^2/2r_0^2}. \quad (17)$$

Because of Eq. (17), the term in the exponent of Eq. (16) reduces to the linear in \mathbf{F} form. Then, considering \mathbf{F} as a random Gaussian variable, the value of $\langle I(\mathbf{r}, t) \rangle$ can be easily obtained in a manner, explained in Ref. [16]. To calculate $\langle I(\mathbf{r}, t) \rangle$, an explicit form of the refractive-index correlation function, $\langle n(\mathbf{r})n(\mathbf{r}') \rangle$, is required. In a statistically homogeneous atmosphere it can be written as

$$\langle n(\mathbf{r})n(\mathbf{r}') \rangle = \int d\mathbf{g} e^{-i\mathbf{g}(\mathbf{r}-\mathbf{r}')} \psi(\mathbf{g}). \quad (18)$$

A widely used the von Karman approximation for the spectrum, $\psi(\mathbf{g})$, is given by

$$\psi(\mathbf{g}) = 0.033 C_n^2 \frac{\exp[-(gl_0/2\pi)^2]}{[g^2 + L_0^{-2}]^{11/6}}, \quad |\mathbf{g}| \equiv g, \quad (19)$$

where the vector \mathbf{g} is defined in the three dimensional domain.

The "source" part of $\langle I^2(\mathbf{r}) \rangle$, given by $\langle b^\dagger b b^\dagger b \rangle$, is approximately equal to $\langle b^\dagger b^\dagger b b \rangle = |\beta|^4$, when the condition $|\beta|^4 \gg |\beta|^2$ is satisfied. This inequality implies that the initial laser radiation is in a multiphoton coherent state. The averaging over independent random quantities \mathbf{a} and \mathbf{a}' can be used instead of the time averaging of the diffuser state. Then we have

$$\begin{aligned} \langle I^2(\mathbf{r}, t) \rangle &= \left| \frac{2\pi\beta^2 r_0^2}{VS} \right|^2 \sum_{\mathbf{q}, \mathbf{k}, \mathbf{q}', \mathbf{k}'} \langle e^{-i\mathbf{k}[\mathbf{r}-\mathbf{c}_\mathbf{q}t + \frac{c}{q_0} \int_0^t dt' t' \mathbf{F}(\mathbf{r}_\mathbf{q}(t'))] - i\mathbf{k}'[\mathbf{r}-\mathbf{c}_{\mathbf{q}'}t + \frac{c}{q'_0} \int_0^t dt' t' \mathbf{F}(\mathbf{r}_{\mathbf{q}'}(t'))]} \\ &\times \left\{ e^{-(Q_\mathbf{a}^2 + Q_{\mathbf{a}'}^2 + \frac{k^2 + k'^2}{4})\frac{r_0^2}{2}} + e^{-[(\mathbf{Q}_\mathbf{a} + \frac{\mathbf{k}}{2})^2 + (\mathbf{Q}_{\mathbf{a}'} - \frac{\mathbf{k}'}{2})^2 + (\mathbf{Q}_{\mathbf{a}'} - \frac{\mathbf{k}}{2})^2 + (\mathbf{Q}_{\mathbf{a}'} + \frac{\mathbf{k}'}{2})^2]\frac{r_0^2}{4}} \right\}. \end{aligned} \quad (20)$$

There are two terms in the braces of Eq. (20). They appear only if the initial four-wave correlation reduces to the pair correlation [16]. Such a modification of the statistical properties of the radiation occurs when the waves

propagate for a long time which is sufficient for randomization of the transverse photon momentum. A more general case, which includes the regime of fast detection, was analyzed in Ref. [28].

The averaging of Eq. (20) over \mathbf{a} and \mathbf{a}' results in

$$\begin{aligned} \langle I^2(\mathbf{r}, t) \rangle &= \left| \frac{2\pi\beta^2 r_1^2}{VS} \right|^2 \sum_{\mathbf{q}, \mathbf{k}, \mathbf{q}', \mathbf{k}'} \langle e^{-i\{\mathbf{k}[\mathbf{r}-\mathbf{c}(\mathbf{q})t] + \mathbf{k}'[\mathbf{r}-\mathbf{c}(\mathbf{q}')t] + \frac{c}{q_0} \int_0^t dt' t' [\mathbf{k}\mathbf{F}(\mathbf{r}(\mathbf{q}, t')) + \mathbf{k}'\mathbf{F}(\mathbf{r}(\mathbf{q}', t'))]\}} \rangle \\ &\times \left\{ e^{-(Q^2+Q'^2)r_1^2/2 - (k^2+k'^2)r_0^2/8} + e^{-[(\mathbf{Q}-\mathbf{Q}')^2 + (\mathbf{k}+\mathbf{k}')^2/4]r_0^2/4 - [(\mathbf{Q}+\mathbf{Q}')^2 + (\mathbf{k}-\mathbf{k}')^2/4]r_1^2/4} \right\}. \end{aligned} \quad (21)$$

In the absence of a phase diffuser, $r_0 = r_1$, the summands in the last braces contribute equally into (21).

Similarly to Eq. (17), the factor $e^{-(Q^2+Q'^2)\frac{r_1^2}{2}}$ in (21) can be expressed in the integral form as

$$e^{-(Q^2+Q'^2)\frac{r_1^2}{2}} = \int \frac{d\mathbf{p}d\mathbf{p}'}{(2\pi r_1^2)^2} e^{i\mathbf{p}\mathbf{Q} + i\mathbf{p}'\mathbf{Q}' - (p^2+p'^2)/2r_1^2}. \quad (22)$$

As we see, the exponent in the left-hand side is represented as a linear form of the force \mathbf{F} . A similar transform is applicable to the second term in the last braces of (21). As a result, the fluctuating force enters the right-hand side of (21) only via the common multiplier, M , given by

$$M = e^{-i \int_0^t dt' \{(\mathbf{p} + \mathbf{k}t'c/q_0)\mathbf{F}[\mathbf{r}(\mathbf{q}, t')] + (\mathbf{p}' + \mathbf{k}'t'c/q_0)\mathbf{F}[\mathbf{r}(\mathbf{q}', t')]\}}. \quad (23)$$

Obtaining of the average value of I^2 reduces to averaging of M with many-fold integration. Assuming the exponent in (23) as a Gaussian random variable, we can write

$$\langle M \rangle = e^{-\frac{1}{2} \left\langle \left(\int_0^t dt' \{(\mathbf{p} + \mathbf{k}t'c/q_0)\mathbf{F}[\mathbf{r}(\mathbf{q}, t')] + (\mathbf{p}' + \mathbf{k}'t'c/q_0)\mathbf{F}[\mathbf{r}(\mathbf{q}', t')]\} \right)^2 \right\rangle} \equiv e^{-\frac{1}{2}(\phi_{PP} + 2\phi_{PP'} + \phi_{P'P'})}. \quad (24)$$

Two types of the correlation functions determine $\langle M \rangle$:

$$\phi_{PP'} = \int_0^t \int_0^t dt' dt'' (\mathbf{p} + \mathbf{k}t'c/q_0) \cdot \langle \mathbf{F}[\mathbf{r}(\mathbf{q}, t')] \mathbf{F}[\mathbf{r}(\mathbf{q}', t'')] \rangle \cdot (\mathbf{p}' + \mathbf{k}'t''c/q_0), \quad (25)$$

$$\phi_{PP} = \int_0^t \int_0^t dt' dt'' (\mathbf{p} + \mathbf{k}t'c/q_0) \cdot \langle \mathbf{F}[\mathbf{r}(\mathbf{q}, t')] \mathbf{F}[\mathbf{r}(\mathbf{q}, t'')] \rangle \cdot (\mathbf{p} + \mathbf{k}t''c/q_0), \quad (26)$$

where symbols P and P' denote sets of three vector variables $P = \{\mathbf{q}, \mathbf{p}, \mathbf{k}\}$ and $P' = \{\mathbf{q}', \mathbf{p}', \mathbf{k}'\}$. The correlation functions of the forces along different ($\mathbf{q} \neq \mathbf{q}'$) and coinciding ($\mathbf{q} = \mathbf{q}'$) trajectories enter Eqs. (25) and (26), respectively. The former can be rewritten as

$$\langle F_\alpha[\mathbf{r}(\mathbf{q}, t')] F_\beta[\mathbf{r}(\mathbf{q}', t'')] \rangle = \langle F_\alpha[\mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'')] F_\beta[0] \rangle, \quad (27)$$

where the notations α and β stand for the x and y - components. The expression for (26) follows from Eq. (27) by setting $\mathbf{q} = \mathbf{q}'$.

The right-hand side of Eq. (27) is assumed to be a function of the coordinate difference, $\mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'')$. It is so if the atmosphere is statistically homogeneous. In the course of averaging, dependence of the coordinate difference on the fluctuating force should be also taken into account. This dependence is given by the relation

$$\begin{aligned} \mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'') &= (\mathbf{e}_z c + \mathbf{c}_{\mathbf{q}'})(t' - t'') - \mathbf{c}_{\mathbf{q} - \mathbf{q}'}(t - t') + \frac{c}{q_0} \int_{t'}^{t''} dt_1 (t' - t_1) \mathbf{F}[\mathbf{r}(\mathbf{q}', t_1)] \\ &+ \frac{c}{q_0} \int_t^{t'} dt_1 (t' - t_1) \{ \mathbf{F}[\mathbf{r}(\mathbf{q}, t_1)] - \mathbf{F}[\mathbf{r}(\mathbf{q}', t_1)] \}, \end{aligned} \quad (28)$$

which follows from Eq. (9). The distance $|\mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'')|$ should be of the order or less than the outer radius, L_0 , of the turbulence. Taking into account that $c \gg |c_{\mathbf{q} - \mathbf{q}'}|, |c_{\mathbf{q}'}|$, we infer that $|t' - t''| \leq L_0/c$. This means that in the right-hand side of Eq. (28) $c_{\mathbf{q}'}$ in the first term and the third term, which is proportional to $(t' - t'')^2$, can be omitted. Then Eq. (28) reduces to

$$\begin{aligned} \mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'') &= \mathbf{e}_z c(t' - t'') - \mathbf{c}_{\mathbf{q} - \mathbf{q}'}(t - t') \\ &+ \frac{c}{q_0} \int_t^{t'} dt_1 (t' - t_1) \{ \mathbf{F}[\mathbf{r}(\mathbf{q}, t_1)] - \mathbf{F}[\mathbf{r}(\mathbf{q}', t_1)] \}. \end{aligned} \quad (29)$$

The last two terms in Eq. (29) describe the displacement of two photons from each other because of the difference of their initial velocities. The term $-\mathbf{c}_{\mathbf{q} - \mathbf{q}'}(t - t')$ describes the divergence of two straight-line trajectories. The last term accounts for the different actions of the atmosphere on the particles moving in different spatial regions.

Obtaining of the average values in Eq. (27), which depend on the wave-vectors $\mathbf{r}(\mathbf{q}, t')$ and $\mathbf{r}(\mathbf{q}', t'')$, seems to be challenging because of the presence of the fluctuating force in $\mathbf{r}(\mathbf{q}, t')$ and $\mathbf{r}(\mathbf{q}', t'')$. Nevertheless the analysis

simplifies if we neglect the correlations between the forces F_α or F_β and the forces entering $\mathbf{r}(\mathbf{q}, t')$ or $\mathbf{r}(\mathbf{q}', t'')$. This simplification can be justified by the following reasonings. The explicit value of the α -force is given by

$$\begin{aligned} F_\alpha[\mathbf{r}(\mathbf{q}, t')] &= F_\alpha[\mathbf{r} - \mathbf{c}_\mathbf{q}(t - t') - \frac{c}{q_0} \int_t^{t'} dt_1 (t_1 - t') F[\mathbf{r}(\mathbf{q}, t_1)]] \\ &= F_\alpha[\mathbf{r}_\perp - \mathbf{c}_{\mathbf{q}_\perp}(t - t') + c\mathbf{e}_z t' - \frac{c}{q_0} \int_t^{t'} dt_1 (t_1 - t') \mathbf{F}[\mathbf{r}(\mathbf{q}, t_1)]] \end{aligned} \quad (30)$$

where the relation $z = ct$ is used.

If the correlation exists, the distance $|\mathbf{r}(\mathbf{q}, t_1) - \mathbf{r}(\mathbf{q}, t')|$ can be estimated by the value $c(t_1 - t') \leq L_0$. In this case, the integral in Eq. (30) is proportional to $(L_0/c)^2$. Hence, the correlation between $F_\alpha[\mathbf{r}(\mathbf{q}, t')]$ and $F[\mathbf{r}(\mathbf{q}, t_1)]$ can be neglected. This approximation implies the physical picture where the variation of the photon momentum on the correlation length, L_0 , is much smaller than q_0 . Therefore, the averaging $\langle F_\alpha F_\beta \rangle$ can be performed in two steps. Firstly, we obtain $\langle F_\alpha F_\beta \rangle$ considering the arguments of F_α and F_β to be fixed. After that, the averaging of the forces, entering the arguments, should be performed. For example, the term (25) is expressed as

$$\phi_{PP'} = \omega_0^2 \int_0^t \int_0^t dt' dt'' \int d\mathbf{g} \psi(g) \mathbf{g} \cdot (\mathbf{p} + \mathbf{k} t' \frac{c}{q_0}) \mathbf{g} \cdot (\mathbf{p}' + \mathbf{k}' t'' \frac{c}{q_0}) \langle e^{-i\mathbf{g}[\mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'')]} \rangle, \quad (31)$$

where the first-step averaging results in appearance of the spectral density $\psi(g)$. The second-step averaging is shown in (31) by the angle brackets. To simplify the derivation of σ^2 , the authors of [16] represented the average of the exponential function in Eq. (31) as a product,

$$\langle e^{-i\mathbf{g}[\mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'')]} \rangle \approx \langle e^{-i\mathbf{g}\mathbf{r}(\mathbf{q}, t')} \rangle \langle e^{i\mathbf{g}\mathbf{r}(\mathbf{q}', t'')} \rangle, \quad (32)$$

neglecting the correlation of the photon displacements $\mathbf{r}(\mathbf{q}, t')$ and $\mathbf{r}(\mathbf{q}', t'')$. Further analysis explains how this correlation can be accounted for.

First of all, it should be noted that we can integrate Eq. (31) over $t' - t''$ because of the presence of the term $\mathbf{e}_z c(t' - t'')$ in $\mathbf{r}(\mathbf{q}, t') - \mathbf{r}(\mathbf{q}', t'')$ [see Eq. (29)]. The corresponding fast oscillating function, $e^{i\mathbf{e}_z \mathbf{g} c(t' - t'')}$, appears in the last factor of Eq. (31). Integration of this factor results in

$$\int_{-\infty}^{\infty} d(t' - t'') e^{i\mathbf{e}_z \mathbf{g} c(t' - t'')} = \frac{2\pi}{c} \delta(g_z). \quad (33)$$

The lower and the upper limits of the integration over $t' - t''$ are replaced by $\mp\infty$. This can be approved when the propagation time, t , is much greater than L_0/c . In other factors in Eq. (31), the substitution $t'' = t'$ is used.

The relation (33) means, that only the $g_{x,y}$ -components enter Eq. (31). In particular, the Fourier-transform $\psi(g)$ should be considered as a function of the two-dimensional vector \mathbf{g}_\perp : $\psi = \psi\left(\sqrt{g_x^2 + g_y^2}\right)$. This observation corresponds to the known Markov approximation [3] where it is assumed that the index-of-refraction fluctuations are delta-function correlated in the direction of propagation. In fact, our derivation, based on the paraxial approximation, supports the validity of the Markov approach which at first sight seems to be doubtful.

Using Eqs. (29) and (33), the expression (31) is simplified to

$$\begin{aligned} \phi_{PP'} = \frac{2\pi\omega_0^2}{c} \int_0^t dt' \int d\mathbf{g} \psi(g) \mathbf{g} \cdot (\mathbf{p} + \mathbf{k}t' \frac{c}{q_0}) \mathbf{g} \cdot (\mathbf{p}' + \mathbf{k}'t' \frac{c}{q_0}) e^{i\mathbf{g}\mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t-t')} \\ \times \langle e^{-i\mathbf{g} \frac{c}{q_0} \int_{t'}^t dt_1 (t_1 - t') \{\mathbf{F}[\mathbf{r}(\mathbf{q}, t_1)] - \mathbf{F}[\mathbf{r}(\mathbf{q}', t_1)]\}} \rangle, \end{aligned} \quad (34)$$

where all the vectors have only the x - and y -components, and $\mathbf{c}_{\mathbf{q}-\mathbf{q}'} = c(\mathbf{q} - \mathbf{q}')/q_0$.

As we see from Eq. (34), to obtain $\phi_{PP'}$ one needs to calculate the average value of the exponential function which is similar to the function in (23). Following the previous procedure, this average can be rewritten as

$$\begin{aligned} & \left\langle \exp \left\{ -i\mathbf{g} \frac{c}{q_0} \int_{t'}^t dt_1 (t_1 - t') \{\mathbf{F}[\mathbf{r}(\mathbf{q}, t_1)] - \mathbf{F}[\mathbf{r}(\mathbf{q}', t_1)]\} \right\} \right\rangle \\ &= \exp \left\{ -2\pi c^3 \int_{t'}^t dt_1 (t_1 - t')^2 \int d\mathbf{g}' \psi(g') (\mathbf{g} \cdot \mathbf{g}')^2 [1 - \langle e^{-i\mathbf{g}' \cdot [\mathbf{r}(\mathbf{q}, t_1) - \mathbf{r}(\mathbf{q}', t_1)]} \rangle] \right\}. \end{aligned} \quad (35)$$

Again, the same function appears in the exponent of the right-hand side of Eq. (35) after using the trajectories (28). Similar steps can be undertaken many times. In this way, the time hierarchy, $0 < t' \leq t_1 \dots \leq t_i \leq t$, is generated. If the photon-turbulence interaction time, $t - t_i$, is short, the disturbance of the trajectory is small and vanishes when $t_i \rightarrow t$. In this case both values, $\mathbf{r}(\mathbf{q}, t_i)$ and $\mathbf{r}(\mathbf{q}', t_i)$, approach the value of \mathbf{r} irrespective of the initial momenta \mathbf{q} and \mathbf{q}' . Therefore we substitute the quantity

$$\frac{1}{2} \langle (\mathbf{g}' \cdot [\mathbf{r}(\mathbf{q}, t_1) - \mathbf{r}(\mathbf{q}', t_1)])^2 \rangle \quad (36)$$

instead of

$$1 - \langle e^{-i\mathbf{g}' \cdot [\mathbf{r}(\mathbf{q}, t_1) - \mathbf{r}(\mathbf{q}', t_1)]} \rangle \quad (37)$$

assuming the exponent in Eq. (37) to be small. The linear in \mathbf{g}' term in the expansion of the exponential factor is ignored because of its zero-value contribution into the integral over \mathbf{g}' in Eq. (35). Then the term (37) reduces to

$$\begin{aligned} & \frac{1}{2} \langle (\mathbf{g}' \cdot [\mathbf{r}(\mathbf{q}, t_1) - \mathbf{r}(\mathbf{q}', t_1)])^2 \rangle \\ & \approx \frac{(t - t_1)^2}{2} (\mathbf{c}_{\mathbf{q}-\mathbf{q}'} \cdot \mathbf{g}')^2 + \frac{\pi c^3}{30} (t - t_1)^5 \int d\mathbf{g}'' \psi(g'') (\mathbf{c}_{\mathbf{q}-\mathbf{q}'} \cdot \mathbf{g}'')^2 (\mathbf{g}' \cdot \mathbf{g}'')^2. \end{aligned} \quad (38)$$

To obtain Eq. (38), the approximate relation,

$$F_\alpha[\mathbf{r}(\mathbf{q}, t_2)] - F_\alpha[\mathbf{r}(\mathbf{q}', t_2)] \approx \mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t_2 - t) \partial_{\mathbf{r}} F_\alpha[\mathbf{r} + \mathbf{c}_{\mathbf{q}}(t_2 - t)], \quad (39)$$

where $t_1 \leq t_2 \leq t$, was used. This approximation is in the spirit of the previous step, where the turbulence effect was assumed as a small perturbation.

Substitution of Eq. (38) into the right-hand side of Eq. (35) and integration over variables \mathbf{g}' , \mathbf{g}'' and t_1 result in

$$\begin{aligned} \exp \left\{ -2.52 \cdot 10^{-3} C_n^2 l_0'^{-7/3} c^3 c_{\mathbf{q}-\mathbf{q}'}^2 (t - t')^5 g^2 \left[1 + \frac{C_n^2 l_0'^{-7/3} c^3 (t - t')^3}{560} + \right. \right. \\ \left. \left. + \frac{\cos 2\theta}{2} \left(1 + \frac{C_n^2 l_0'^{-7/3} c^3 (t - t')^3}{2 \cdot 560} \right) \right] \right\}, \end{aligned} \quad (40)$$

where $l_0' = l_0/2\pi$, and θ is the angle between the two-dimensional vectors \mathbf{g} and $\mathbf{q} - \mathbf{q}'$.

After substitution of (40) into (35), (35) into (34) and (34) into (25), we calculate $\langle I^2(\mathbf{r}, t) \rangle$. Many-fold integrations over the variables \mathbf{q} , \mathbf{q}' , \mathbf{p} , \mathbf{p}' , \mathbf{k} , \mathbf{k}' , θ , and t' are performed mainly numerically with employing a computer cluster. In the course of integration, we have used the Tatarskii modification of the refractive index spectrum which is derived from the von Karman form (19) by setting $L_0^{-1} = 0$. The results for σ^2 are shown in Figs. 1-3.

4 Discussion

Figs. 1-3 can be used to illustrate the importance of the correlations of different trajectories. To simplify our argumentations, we consider a coherent

laser beams, i.e., the case $r_0 = r_1$. Two terms in the last braces of Eq. (21) contribute equally into $\langle I^2(\mathbf{r}, t) \rangle$. Moreover, if one sets $\phi_{PP'} = 0$ in Eq. (24), thus ignoring the correlations of photons with different initial momenta, we obtain $\langle I^2(\mathbf{r}, t) \rangle = 2\langle I(\mathbf{r}, t) \rangle^2$. The scintillation index, σ^2 , is equal to unity here. This physical picture is realized for a long-distance propagation

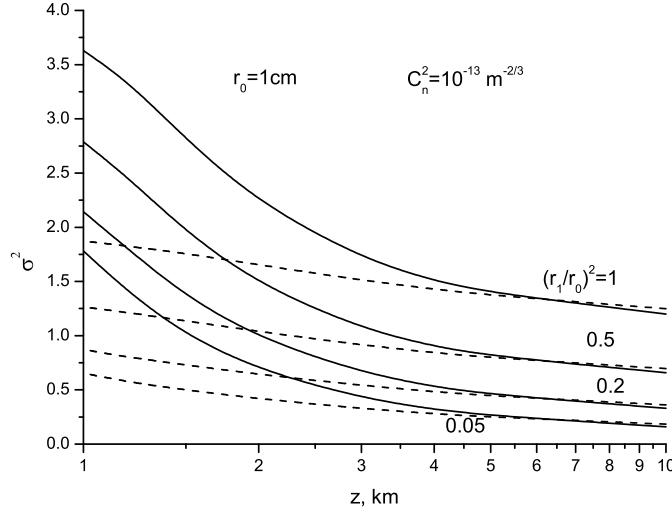


Figure 1: Scintillation index of a coherent and partially coherent beams in the atmosphere versus propagation distance z . Dashed curves correspond to the multiplicative approximation (32) for the photon correlations; solid curves are obtained within the present paper's approach [see Eqs. (35 -40)]. $C_n^2 = 10^{-13} m^{-2/3}$, $r_0 = 0.01 m$, $\frac{l_0}{2\pi} = 10^{-3} m$, and $q_0 = 10^7 m^{-1}$. The upper two curves correspond to the coherent beam.

($t \rightarrow \infty$) when the oscillating factor $e^{i\mathbf{g}\mathbf{c}\mathbf{q}-\mathbf{q}'(t-t')}$ confines the effective volume of the integration over \mathbf{g} and $\mathbf{q} - \mathbf{q}'$ to zero [see Eq. (34)]. For finite values of t , the contribution of $\phi_{PP'}$ becomes quite sizeable that is seen in Figs. 1-3 where the values of σ^2 are greater than unity. There is a positive contribution of $\phi_{PP'}$ term into the last exponent in Eq. (24) when the vectors \mathbf{p}, \mathbf{k} and \mathbf{p}', \mathbf{k}' have opposite signs and the difference $|\mathbf{q} - \mathbf{q}'|$ is not too large. The most favorable conditions are realized when

$$\mathbf{p} = -\mathbf{p}', \quad \mathbf{k} = -\mathbf{k}', \quad \mathbf{q} = \mathbf{q}'. \quad (41)$$

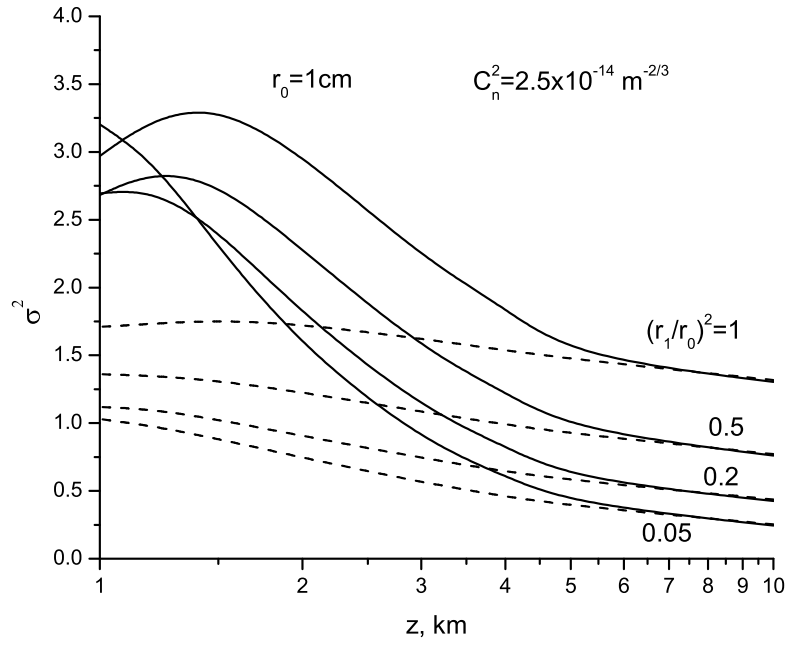


Figure 2: The same as in Fig. 1 but for a weaker turbulence strength: $C_n^2 = 2.5 \times 10^{-14} \text{ m}^{-2/3}$.

In this case the sum $\phi_{PP} + 2\phi_{PP'} + \phi_{P'P'}$ is equal to zero. Eqs. (41) can be interpreted as the "super-correlation" conditions under which the value of M is equal to unity and does not depend on the turbulence.

The dependence of σ^2 on the initial radius r_0 can be explained as follows. The characteristic values of the initial momentum, $\tilde{q} \sim \sqrt{2}/r_0$, is greater for small r_0 . Hence the volume of integration over $\mathbf{q} - \mathbf{q}'$ is also greater. At the same time the corresponding increase of $\phi_{PP'}$ occurs only for short distances, z , where time intervals t are sufficiently small and the oscillating factor in Eq. (34) is close to unity. Therefore, when r_0 decreases, there is an increase of σ^2 accompanied with the displacement of the region with enhanced fluctuations towards small z . This is clearly seen in Fig. 3.

In a similar way we can explain a considerable difference of σ^2 found for the plane-wave and spherical-wave models of radiation in Ref. [34] (Figs. 1 and 2 there). It follows from the above reasonings that this effect arises due to very different initial \mathbf{q} -volumes in the two models.

Also, the calculations of σ^2 in the Ref. [15] should be mentioned where a simplified model of the turbulence was used (see Fig. 1 there). The results of Ref. [15] well correlate with ours.

Comparing the results of the present paper and those, based on the approximation of uncorrelated trajectories (32) (respectively, solid and dashed lines in Figs. 1 and 2), we see a more pronounced growth of σ^2 at a moderate turbulence in the former case. Figures 1-3 illustrate that this holds true for the distances of 1 – 3 km. We attribute the evident distinction of the results to a better accuracy of accounting for the correlations of the photon trajectories. At the same time, both approaches provide the known in the literature saturation effect: $\sigma^2 \rightarrow r_1^2/r_0^2$ when $z \rightarrow \infty$.

The phase diffuser with a short characteristic time (a high-frequency diffuser) does not change qualitatively the physical picture described above. At the same time, both approaches reveal an ability of the diffuser to suppress scintillations which is favorable for communication performances.

The effect of the phase diffuser is explained as follows. The initial phase relief, introduced by the diffuser, varies in time. The photon trajectories depend on the initial state of the radiation and varies synchronously with the diffuser state. A "slow" detector integrates the contribution of these photons. Although the atmosphere stays almost frozen during the integration time, the diffuser provides a better averaging of the propagating radiation over the refractive-index relief. Therefore, the fluctuations of the detected signal decrease.

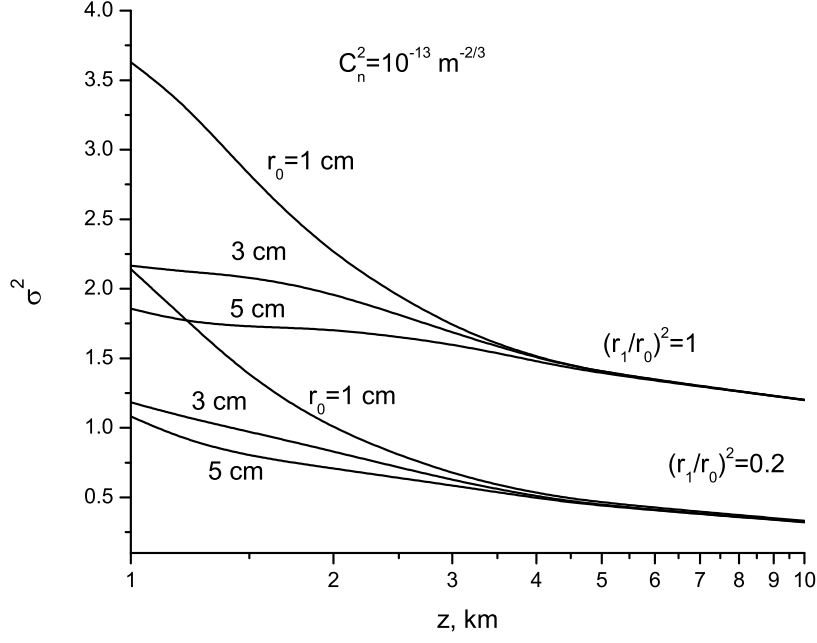


Figure 3: Scintillation index versus propagation distance z for different initial radii of the beam: $r_0 = 0.01\text{m}$, 0.03m , 0.05m . The rest of the parameters are the same as in Fig 1.

This is not a unique way to suppress fluctuations. For example, the authors of Ref. [35] proposed to use asymmetric optical vortices. The range of a weak and moderate turbulence was studied. Numerical simulations of the beam propagation showed promising results. It should be emphasized that in this case the experimental setup does not require a high-frequency phase diffuser.

5 Applicability of the distribution function approach for short distances

Our analysis is based on Eq. (21) obtained within the concept of photon trajectories. To consider photons as particles, whose density in the (\mathbf{r}, \mathbf{q})

domain is defined by the distribution function $f(\mathbf{r}, \mathbf{q}, t)$, the uncertainty of the momentum, \mathbf{q} , should be small. The value of the uncertainty can be estimated from the definition of the distribution function (1) as $\tilde{k}/2$. It follows from Eq. (14) that close to the source and in the absence of the diffuser the ratio $\frac{\tilde{q}}{\tilde{k}/2} \sim \frac{\langle q^2 \rangle^{1/2}}{\langle k^2/4 \rangle^{1/2}} = \frac{(2/r_0^2)^{1/2}}{(2/r_0^2)^{1/2}} = 1$. Hence in the vicinity of the source, our calculations of σ^2 are not applicable if the light is in a coherent state.

The situation changes drastically for a remote detector. With increase of the propagation path, z , the value of \tilde{q} increases. The corresponding gain of the photon momentum, $\Delta\mathbf{q}$, is generated by a random force, \mathbf{F} . Hence the average value, $\langle \Delta\mathbf{q} \rangle$, is equal to zero while the nonzero mean-square value is given by [16]

$$\langle \Delta q^2 \rangle = 0.066\pi^2 \Gamma(1/6) q_0^2 l_0'^{-1/3} C_n^2 z. \quad (42)$$

In contrast to \tilde{q} , the value of \tilde{k} decreases because of the broadening of the beam. The mean-square of the beam radius is given by [6, 16]

$$R^2 = \frac{r_0^2}{2} \left[1 + \frac{4z^2}{q_0^2 r_0^2 r_1^2} + \frac{8z^3 T}{r_0^2} \right], \quad (43)$$

where $T = 0.558 l_0'^{-1/3} C_n^2$. When the last term in square brackets dominates, the ratio $\tilde{q}^2/(\tilde{k}/2)^2$ can be estimated as

$$\langle \Delta q^2 \rangle R^2 \approx 15 \cdot q_0^2 l_0'^{-2/3} C_n^4 z^4, \quad (44)$$

where $\langle \Delta q^2 \rangle$ is assumed to be of the order of \tilde{q}^2 thus ignoring the square of the initial momentum $2/r_0^2$.

Substituting $z = 10^3 m$, $q_0 = 10^7 m^{-1}$, and $l_0 = 2\pi \cdot 10^{-3} m$ into Eq. (44), we obtain $\frac{\tilde{q}}{\tilde{k}/2} \sim 21$ that provides adequacy of our approach for the whole range of z variations shown in Figs. 1-3. This range concerns not only coherent, but also partially coherent beams. For partially coherent beams, the minimum z can be even smaller than for coherent beams. This is because of an additional diffuser-caused growth of \tilde{q}^2 , which is estimated by the value $\Delta \tilde{q}_{diffuser}^2 \sim 2/r_1^2$.

6 Conclusion

The paper continues the studies presented in Refs. [8, 16]. Using the approach of the distribution function, the problem of obtaining of σ^2 reduces to

calculation of the correlations between different photon trajectories. Assuming the outer radius of turbulent eddies much smaller than the propagation distance, the iterative procedure for calculations of these correlations is developed. The modified approach makes it possible to extend applicability of the theory to a wider range of the propagation distances. This range includes a strong turbulence as well as a considerable part of a moderate turbulence where the scintillation index tends to reach its maximum value. The criterion, derived in Sec. 5, imposes the restriction on our theory from the side of short distances (weak turbulences).

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